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# Two-dimensional ideal magnetohydrodynamics and differential geometry

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**Abstract.** It is shown that equations of two-dimensional ideal magnetohydrodynamics may be regarded as geodesic equations on appropriate infinite dimensional Lie group. The physical interpretation of such a geometric picture is given using an analogy with two-and-a-half-dimensional ideal hydrodynamics. The sectional curvature responsible for the separation of neighbouring geodesics and, hence, for stability is calculated.

It is well known that a universal construction exists which allows to associate to arbitrary Lie group $G$ with a Lie algebra $g$ a quadratically nonlinear dynamical system on the dual space $g^*$, once a mapping $g \rightarrow g^*$ (a metric) is introduced. The resulting generalized Euler equations read

$$\dot{\omega}_i + g^{jl} C_{ij}^k \omega_k \omega_l = 0 \tag{1}$$

where $\omega_i \in g^*, i = 1, 2, \ldots;$ $C_{ij}^k$ are the structure constants of $g$ in some basis and $g^{ik}$ are the components of the metric (a summation over repeated indices is assumed hereafter). The well- known examples of this construction are a rigid body motion and its generalizations from original group of three-dimensional rotations to an arbitrary classical Lie group [1, 2]. The most transparent infinite-dimensional example is that of the hydrodynamics of an ideal incompressible fluid (IHD) when the group is a group of volume-preserving diffeomorphisms of a domain of some fluid flow. In the case of periodic boundary conditions (BC) the indices in (1) are just the Fourier-indices. It is also known [1] that equation (1) admits another reading. Namely, by right- (left-) hand shifts the metric may be transported to an arbitrary point of a group manifold, hence providing a Riemannian structure for the latter. Then equation (1) is equivalent to a geodesic equation for this manifold and, in this way, it aquires a variational meaning. In physical terms these two descriptions of the same system correspond to Eulerian and Lagrangian viewpoints, respectively.

The co-adjoint orbit (i.e. a manifold obtained from arbitrary point belonging to $g^*$ by all possible group transformations) serves as a phase space and provides a symplectic (Hamiltonian) structure for (1), so its geometry is crucial for Eulerian stability analysis [1, 3], while a Riemannian geometry of the group as a whole comes into play when the question of Lagrangian stability is addressed. For example [1], the curvature of the group is responsible for a behaviour of the neighbouring geodesics.

In the context of fluid dynamics, besides the IHD itself, a number of fundamental systems have the form (1). In particular, it is known [3, 4] that equations of two-dimensional ideal magnetohydrodynamics (IMHD) in terms of vorticity and magnetic potential are of the form (1) with structure constants being those of a semi-direct product of symplectic diffeomorphisms of $\mathcal{D}$ and functions on $\mathcal{D}$, where $\mathcal{D}$ denotes a flow domain.

In the present paper our purpose is to apply a 'geodesic' approach, first announced in [5], for 2D IMHD. First, we shall try to understand the physical meaning of such a description. We would like to remind the reader that in the case of IHD the geodesics lie in the space of volume-preserving diffeomorphisms and the group coordinates are identified with the Lagrangian coordinates of the fluid particles (modulo the incompressibility constraint) [1, 5, 6]. However, in the present case the Lagrangian coordinates in a plane are not sufficient, since we can describe only a subgroup (namely, symplectic diffeomorphisms) in this way and not the group as a whole.

The equations of ideal (no viscosity, infinite conductivity) incompressible 2D MHD with magnetic field in the plane result from the full three-dimensional (3D) equations

$$\begin{aligned} &\dot{\boldsymbol{v}} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} - \boldsymbol{h}\cdot\nabla\boldsymbol{h} + \nabla p^* = 0 \\ &\dot{\boldsymbol{h}} - \nabla\times(\boldsymbol{v}\times\boldsymbol{h}) = 0 \\ &\nabla\cdot\boldsymbol{v} = \nabla\cdot\boldsymbol{h} = 0 \end{aligned} \tag{2}$$

when one supposes that $\boldsymbol{v}$ and $\boldsymbol{h}$ are 2D vectors independent (as well as $p^*$) of the third coordinate $z$. Introducing the magnetic potential $a$, streamfunction $\psi$ and vorticity $\omega$

$$\boldsymbol{v} = \operatorname{sgrad}\psi \qquad \boldsymbol{h} = \operatorname{sgrad} a \qquad \omega = -\Delta\psi \tag{3}$$

one obtains ($J$ denoting a Jacobian)

$$\begin{aligned} &\dot{\omega} + J(\omega,\psi) - J(a,\Delta a) = 0 \\ &\dot{a} + J(a,\psi) = 0\,. \end{aligned} \tag{4}$$

We would like to emphasize a rather obvious fact that although the system (4) is formally two-dimensional, it nevertheless describes a three-dimensional physics. Indeed, from Ampere's law $\nabla\times\boldsymbol{h} = \boldsymbol{j}$ and the fact that $\boldsymbol{h}$ is a vector lying in the $x$–$y$ plane, it follows that there is a non-vanishing ( but $z$-independent) component of an electric current $\boldsymbol{j}$ in the $z$-direction orthogonal to the plane. The infinitesimal group action on the fields $\omega$ and $a$ is given by the following transformation [4]:

$$\begin{aligned} &\delta\omega = J(\chi,\omega) + J(\sigma,a) \\ &\delta a = J(\chi,a) \end{aligned} \tag{5}$$

here $\chi$ and $\sigma$ are arbitrary smooth functions—infinitesimal parameters of the transformation. The structure constants may be extracted from the commutator of these transformations, and equations (4) are of the form (1) if an energy (kinetic + magnetic) functional

$$H = \frac{1}{2}\int_{\mathcal{D}} (\omega\psi - a\Delta a)\,\mathrm{d}x\,\mathrm{d}y \tag{6}$$

is chosen as a metric. Assuming for simplicity periodic BC and zero mean values for vorticity and magnetic potential we may invert the Laplacian

$$\psi = -\Delta^{-1}\omega\,.$$

Since the metric, by definition, relates $g^*$ and $g$, i.e. the co-adjoint and adjoint representations, the streamfunction $\psi$ and the current $j = -\Delta a$ belong to the latter and transform according to the following formulas:

$$\begin{aligned} \delta\psi &= J(\chi, \psi) \\ \delta j &= J(\chi, j) + J(\sigma, \psi)\,. \end{aligned} \tag{7}$$

The finite transformations are given by an exponentiation of (5) or (7) and parameters of these finite transformations are the coordinates on the group manifold. These transformations were obtained in [4] using the fundamental representation of the group realized on Clebsch-like variables.

Now if we want, according to the aforementioned general philosophy, to consider equation (4) as a geodesic equation on the group manifold, the question arises what is the meaning of a coordinate related to the $\sigma$-transformation in (5), (7). As to the $\chi$-transformation, its meaning is clear: it corresponds to an area-preserving change of variables, and the corresponding group coordinate is a Lagrangian coordinate of the fluid particle on the plane (of the two Lagrangian coordinates $X(x, y)$, $Y(x, y)$ only one is independent due to incompressibility). To answer this question we shall use a realization of the group action different from the one used in [4]. It is based on the above-mentioned fact that 2D IMHD is in fact 'two-and-a-half'-dimensional. We start from a simple heuristic argument. Take the 3D Euler equations

$$\dot{\boldsymbol{V}} + \boldsymbol{V} \cdot \nabla \boldsymbol{V} + \nabla p = 0 \qquad \nabla \cdot \boldsymbol{V} = 0 \tag{8}$$

where $\boldsymbol{V}$ is a 3D vector $(v_1, v_2, W)$, and suppose that nothing depends on $z$-coordinate. The equations of two-and-a-half-dimensional IHD follow

$$\begin{aligned} &\dot{\boldsymbol{v}} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \nabla p = 0 \\ &\dot{W} + \boldsymbol{v} \cdot \nabla W = 0 \\ &\nabla \cdot \boldsymbol{v} = 0 \end{aligned} \tag{9}$$

where all the vectors are now two-dimensional. This is, of course, equivalent to

$$\begin{aligned} &\dot{\omega} + J(\omega, \psi) = 0 \\ &\dot{W} + J(W, \psi) = 0\,. \end{aligned} \tag{10}$$

The kinetic energy is

$$H = \frac{1}{2} \int_D (\omega\psi + W^2)\, \mathrm{d}x\, \mathrm{d}y\,. \tag{11}$$

The similarity between (10) and (4) becomes obvious once one rewrites the first equation in (10) as

$$\dot{\omega} + J(\omega, \psi) - J(W, -W) = 0\,. \tag{12}$$

The only difference between the two systems is a choice of the metric: $-\Delta \to 1$ in the second term in the Hamiltonian. The conclusion that the group structure is the same for both systems follows also from the fact that applying (with an obvious change $a \to W$) the magnetohydrodynamical Poisson bracket [4]

$$\{A[\omega, a], B[\omega, a]\} = -\int_D \left[\omega J\left(\frac{\delta A}{\delta\omega}, \frac{\delta B}{\delta\omega}\right) + a\left[J\left(\frac{\delta A}{\delta\omega}, \frac{\delta B}{\delta a}\right) + J\left(\frac{\delta A}{\delta a}, \frac{\delta B}{\delta\omega}\right)\right]\right] \mathrm{d}x\, \mathrm{d}y \tag{13}$$

to the Hamiltonian (11) one obtains equations (10). However, the Poisson bracket is a Kirillov bracket and is completely defined by the Lie algebra. Therefore, these latter are the same for both systems. Yet another demonstration comes from the fact that if one takes a Lie algebra of 3D divergenceless vector fields, which is defined by the commutator

$$[V, U] = V \cdot \nabla U - U \cdot \nabla V \tag{14}$$

and restricts these fields to having the form

$$V = (v, W) \qquad \nabla \cdot v = 0 \qquad v = v(x, y) = (v_1, v_2) \qquad W = W(x, y)$$

(the operator nabla in this formula is two-dimensional), a straightforward calculation shows that one obtains a subalgebra of the initial algebra, which coincides with the algebra of infinitesimal transformations (7), where $j$ corresponds to $W$ and $\psi$ is a streamfunction of $v$. As was mentioned before, the group of 3D volume-preserving diffeomorphisms which has equation (14) as its Lie algebra is parametrized by Lagrangian coordinates of fluid particles $X(x, y, z), Y(x, y, z), Z(x, y, z)$ (subject to the incompressibility constraint). In addition, the subgroup in question is parametrized by these coordinates, but now they are $z$-independent.

Returning to 2D IMHD, we see that as its group manifold is the same it may be described by the same coordinates, namely a shift in the new coordinate (with respect to Lagrangian coordinates on the plane) is generated by the current $j$ in the same way as a shift in $z$ is generated by the vertical velocity $W$. Therefore, this new coordinate is related to a charge transport in the system. However, although the coordinates are the same the Riemannian structure and, as a consequence, the behaviour of the geodesics are different, owing to the difference in metrics.

Let us turn now to technical aspects of the 'geodesic' description of IMHD. At least on the heuristic level, a formalism developed in the classical paper [1] may be applied to an arbitrary infinite-dimensional Lie group. Once structure constants and a metric are given, one may calculate a curvature tensor and, hence, determine a divergence of the neighbouring geodesics. But, unlike the case of IHD where separation of geodesics is directly related to a separation of fluid particles [6], the geodesics now have a new degree of freedom and may diverge in the new dimension related to charge transport.

Let us consider, following [1], a Riemannian structure generated by some right-invariant metric on a Lie group. Suppose that the structure constants in a basis formed by vectors $e_A$ are $C^C_{AB}$

$$[e_A, e_B] = C^C_{AB} e_C \qquad e_A \in g \quad A = 1, 2, \ldots . \tag{15}$$

Introducing in the Lie algebra a metric defined by the tensor $g_{AB}$

$$\langle e_A, e_B \rangle = g_{AB} \tag{16}$$

one may obtain a Riemannian structure on the group considering a right- invariant metric, defined on the right-invariant vector fields corresponding to $e_A$. A covariant derivative compatible with this metric may be calculated in the vicinity of the unit element of the group

$$\begin{aligned} \nabla_{e_A} e_B \equiv \nabla_A e_B = \Gamma^C_{AB} e_C \\ \Gamma^C_{AB} = \tfrac{1}{2}(C^C_{AB} - g^{CD} C^E_{AD}\, g_{EB} - g^{CD} C^E_{BD}\, g_{EA}) \,. \end{aligned} \tag{17}$$

Here $\Gamma^C_{AB}$ are the coefficients of the symmetric connection. The components of a curvature tensor are

$$R_{KLMN} = -\langle \nabla_K \nabla_L e_M - \nabla_L \nabla_K e_M - \nabla_{[e_K, e_L]} e_M, e_N \rangle \,. \tag{18}$$

Consider now the IMHD case. An element of the Lie algebra has two components (each being in turn an infinite-dimensional vector): a hydrodynamic one (a streamfunction) and a magnetic one (a current). Let us denote Fourier-components of the corresponding basis vectors as $e_n$ ('non-primed') and $e_{n'}$ ('primed'). To simplify the notation, we shall use hereafter the small italic letters $m, n, \ldots; m', n', \ldots$ to denote vector Fourier-indices: $n \equiv (n_1, n_2)$; $n \times m \equiv n_1 m_2 - n_2 m_1$. For the structure constants in this basis we have

$$\begin{aligned} C^k_{nm} &= n \times m\ \delta(k - n - m) \\ C^{k'}_{nm'} &= n \times m'\ \delta(k' - n - m') \\ C^k_{nm'} &= C^k_{n'm'} = C^{k'}_{n'm'} = 0\,. \end{aligned} \tag{19}$$

The components of the metric tensor follow from (6)

$$\begin{aligned} g_{nm} &= n^2\delta(n+m) \qquad g_{n'm'} = n'^{-2}\delta(n'+m') \qquad g_{nm'} = 0 \\ g^{nm} &= n^{-2}\delta(n+m) \qquad g^{n'm'} = n'^2\delta(n'+m') \qquad g^{nm'} = 0\,. \end{aligned} \tag{20}$$

From (17), (19), (20) we obtain

$$\begin{aligned} \nabla_n e_m &= \frac{n \times m\ (n+m)\cdot m}{(n+m)^2}\, e_{n+m} \\ \nabla_{n'} e_{m'} &= \frac{n' \times m'}{2(n'+m')^2}\left(\frac{1}{m'^2} - \frac{1}{n'^2}\right) e_{(m'+n')} \\ \nabla_n e_{m'} &= \frac{n \times m'}{2}\left[1 + \frac{(n+m')^2}{m'^2}\right] e_{(n+m')'} \\ \nabla_{m'} e_n &= \frac{m' \times n}{2}\left[1 - \frac{(n+m')^2}{m'^2}\right] e_{(n+m')'} \end{aligned} \tag{21}$$

where $e_{m'+n'}$ on the right-hand side of the second equation is a *non-primed* basis vector. We also give the corresponding formulas for the case of two-and-a-half-dimensional IHD

$$\begin{aligned} g_{nm} &= n^2\delta(n+m) \qquad g_{n'm'} = \delta(n'+m') \qquad g_{nm'} = 0 \\ g^{nm} &= n^{-2}\delta(n+m) \qquad g^{n'm'} = \delta(n'+m') \qquad g^{nm'} = 0 \end{aligned} \tag{22}$$

$$\begin{aligned} \nabla_n e_m &= \frac{n \times m\ (n+m)\cdot m}{(n+m)^2}\, e_{n+m} \\ \nabla_{n'} e_{m'} &= 0 \qquad \nabla_n e_{m'} = n \times m'\ e_{(n+m')'} \qquad \nabla_{m'} e_n = 0\,. \end{aligned} \tag{23}$$

Now we are able to calculate a curvature tensor, according to (18). The 'hydrodynamic' components $R_{klmn}$ of this tensor are, evidently, the same as in 2D IHD (cf [1]). However, new contributions appear. Let us calculate, for example, a mixed-component $R_{k'lm'n}$

$$\begin{aligned} R_{k'lm'n} &= -\langle \nabla_{k'}\nabla_l e_{m'} - \nabla_l \nabla_{k'} e_{m'} - \nabla_{[e_{k'}, e_l]} e_{m'}, e_n \rangle \\ &= \langle \nabla_l e_{m'}, \nabla_{k'} e_n \rangle - \langle \nabla_{k'} e_{m'}, \nabla_l e_n \rangle + C^{r'}_{k'l}\langle \nabla_{r'} e_{m'}, e_n \rangle\,. \end{aligned} \tag{24}$$

Consider first the case of two-and-a-half-dimensional IHD. Since all the covariant derivatives in 'primed' directions are zero (which corresponds to the fact that vertical and horizontal velocities do not interact in eqs. (10)), all the mixed components of the curvature tensor vanish. Thus, the group manifold is flat in all sections containing a 'primed' direction.

Consider now the IMHD case. All three terms in (24) are non-vanishing. The first one gives

$$\langle \nabla_l e_{m'}, \nabla_{k'} e_n \rangle = \frac{1}{4}\delta(k'+l+m'+n)(l\times m')\left(1+\frac{(l+m')^2}{m'^2}\right)\frac{1}{(l+m')^2}(k'\times n)\left(1-\frac{(k'+n)^2}{k'^2}\right) \equiv -\delta(k'+l+m'+n) f^+_{lm'} f^-_{nk'} \tag{25}$$

where

$$f^{\pm}_{lm'} = \frac{1}{2}(l\times m')|l+m'|\left(\frac{1}{(l+m')^2} \pm \frac{1}{m'^2}\right). \tag{26}$$

The second term gives

$$\langle \nabla_{k'} e_{m'}, \nabla_l e_n \rangle = (k'\times m')(l\times n)\frac{(l+n)\cdot n}{2(l+n)^2}\left(\frac{1}{m'^2}-\frac{1}{k'^2}\right)\delta(k'+l+m'+n). \tag{27}$$

The third term in (24) is

$$C^{r'}_{k'l}\langle \nabla_{r'} e_{m'}, e_n\rangle = -\frac{(k'\times l)(n\times m')}{2}\left(\frac{1}{m'^2}-\frac{1}{(m'+n)^2}\right)\delta(k'+l+m'+n). \tag{28}$$

Now let us calculate the curvature of a section defined by the current $j = \cos k'\cdot x$ (all non-primed components are zero) and arbitrary streamfunction (all primed components are zero), i.e by the pair of vectors

$$\xi = \frac{1}{2}(e_{k'} + e_{-k'}) \qquad \eta = \sum_l x_l e_l . \tag{29}$$

To do this we need the following combination:

$$K(\xi,\eta) = R_{k'lm'n}\xi_{k'}\eta_l\xi_{m'}\eta_n = \frac{1}{4}\sum_l [\tilde{R}_{k',l,k',-2k'-l}x_l x_{-2k'-l} + \tilde{R}_{-k',l,-k',2k'-l}x_l x_{2k'-l} + \tilde{R}_{k',l,-k',-l}x_l x_{-l} + \tilde{R}_{-k',l,k',-l}x_l x_{-l}] . \tag{30}$$

Here

$$R_{k'lm'n} \equiv \tilde{R}_{k'lm'n}\delta(k'+l+m'+n) .$$

The contribution (27) is proportional to $(k'\times m')$, so it does not give rise to (30) where only the terms with $m' = \pm k'$ enter. Using equations (25), (28) and the fact that

$$f^+_{l,-k'} f^-_{l,-k'} = f^+_{l-2k',k'} f^-_{l-2k',k'} \tag{31}$$

we obtain for the sectional curvature (the calculation is similar to one given in lemma 11 of [1], for which we deliberately kept similar notation)

$$K(\xi,\eta) = -\frac{1}{16}\sum_l \left[(l\times k')^2(l+k')^2\left(\frac{1}{(l+k')^4}-\frac{1}{k'^4}\right) - 2(l\times k')^2\left(\frac{1}{k'^2}-\frac{1}{(l+k')^2}\right)\right]|x_l - x_{l+2k'}|^2 = \frac{1}{16}\sum_l (l\times k')^2\frac{[(l+k')^2+k'^2]^2-(2k'^2)^2}{k'^4(l+k')^2}|x_l - x_{l+2k'}|^2 . \tag{32}$$

It follows from this formula that the curvature for the section (29) is positive for perturbations with high enough wavenumbers (if $|l| > 2|k'|$) but may be negative for low wavenumber ones. This means that the 'purely magnetic' geodesic $\xi$ (29) does not exhibit an instability with respect to high-wavenumber perturbations of 'purely hydrodynamic' type. On the contrary, all the sections containing a flow of this form in pure IHD have negative curvature and the geodesic is absolutely unstable [1].

In conclusion, we have shown that analogously to the case of IHD the IMHD admits a geodesic interpretation. In the present paper we have limited ourselves to the 2D case. There are two major reasons for this. First, there is a certain difference between hydrodynamics in even and odd dimensions [7], so, although it is clear that three-dimensional MHD admits a geometric interpretation as well, it will be somewhat different. Second, an algebraic structure analogous to that of 2D IMHD appears in other physically meaningful (quasi-) bidimensional systems: a Boussinesque stratified fluid [8] and axisymmetric flows with swirl [9]. Of course, we report here a first step in the 'geometric' approach to MHD; the physical consequences of the positiveness or negativeness of the curvature of the section containing a given flow need to be investigated more thoroughly. We plan to do this in a subsequent publication.

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